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Superconvergence and Duality [1]

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Abstract

Superconvergence relations for the transverse gauge field propagator can be used in order to show that the corresponding gauge quanta are not elements of the physical state space, as defined by the BRST algebra. With a given gauge group, these relations are valid for a limited region in the number of matter fields, indicating a phase transition at the boundary. In the case of SUSY gauge theories with matter fields in the fundamental representation, the results predicted by superconvergence can be compared directly with those obtained on the basis of duality and the conformal algebra. There is exact agreement.

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SUPERCONVERGENCE AND DUALITY. Superconvergence of the gauge field propagator leads to general arguments for the absence of transverse gauge field excitations from the physical state space of the theory. These arguments are valid for SUSY and for non-SUSY theories. The presence of superconvergence relations depends upon the gauge group and the the number and typ of matter fields. Duality is another important concept which makes statements about the phase structure of SUSY gauge theories. For SUSY theories of interest, these quite different methods give exactly the same results.
It is reasonable to consider $N = 1$ SUSY theories with the gauge group $SU(N_C)$ and N_F massless quark fields in the fundamental representation. Generalizations will be discussed later. In the Landau gauge, and for appropriate values of N_F, there exists the SUPERCONVERGENCE RELATION [1]

$$\int_{-0}^{\infty} dk^2 \rho(k^2, \kappa^2, g) \quad = \quad 0 \ , \tag{1}$$

where $\kappa^2 < 0$ is the normalization point. The integrand ρ is the discontinuity of the structure function for the transverse gauge field propagator. It represents the norm of the states $\tilde{A}^{\mu\nu}(k)|0\rangle$, where $A^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$, and A^ν is the component gauge field of the corresponding superfield. Explicitely,

$$\begin{aligned} \langle 0|\tilde{A}_a^{\mu\nu}(k')\tilde{A}_b^{\varrho\sigma}(-k)|0\rangle \quad &= \quad \delta_{ab}\theta(k^0)\delta(k'-k)\pi\rho(k^2) \\ \times(-2)(2\pi)^4 \left(k^\mu k^\varrho g^{\nu\sigma} - k^\mu k^\sigma g^{\nu\varrho} + k^\nu k^\sigma g^{\mu\varrho} - k^\nu k^\varrho g^{\mu\sigma}\right) \quad . \end{aligned} \tag{2}$$

With test functions $C^a_{\mu\nu}(k)$, one can form states
$\Psi(C) = \int d^4k C^a_{\mu\nu}(k)\tilde{A}_a^{\mu\nu}(-k)|0\rangle$, and obtain the norm

$$\begin{aligned} (\Psi(C), \Psi(C)) \quad &= \quad \int d^4k\theta(k^0)\pi\rho(k^2)C(k) \ , \\ C(k) \quad &= \quad -8(2\pi)^4 k^\mu \overline{C}^a_{\mu\nu}(k) k^\rho C^a_{\rho\sigma} g^{\nu\sigma} \ , \end{aligned} \tag{3}$$

where $C(k) > 0$ for $k^2 \geq 0,\ k^0 \geq 0$. For positive values of the gauge parameter α, there is constant term α/α_0 on the righthand side of Eq.(1), where $\alpha_0 = -\gamma_{00}/\gamma_{01}$. Here $\gamma(g^2, \alpha) = (\gamma_{00} + \alpha\gamma_{01})g^2 + \cdots$ is the anomalous dimension of the gauge field. But for the problem of phase transitions, it is sufficient to work in the Landau gauge. For values of N_F and N_C for which the superconvergence relation is valid, one can show that the transverse gauge field quanta must be confined in the sense that they are not elements of the physical state space of the theory. The physical state

space is defined in terms of a BRST cohomology, and is invariant under Poincaré and equivalence transformations. In the space with indefinite metric, it requires a detailed analysis of the states $\tilde{A}^{\mu\nu}(k)|0\rangle$ in order to proof this fact. One uses invariant projections into the subspace of positive energy states, and proceeds to isolate the subset of physical states. A complete discussion may be found in [2]. There are no superconvergence relations for the quark propagator. But for the typ of theories considered, the absence of all gluon states from the physical subspace should also imply the absence of quark states associated with the same gauge group [2,3].

The values of N_F, for which the superconvergence relation (1) is valid, are obtained from the leading asymptotic term of the gauge field structure function $D(k^2, \kappa^2, g)$ for $k^2 \to 0$ in all directions of the complex plane. This term is given by

$$- k^2 D(k^2, \kappa^2, g) \;\; \simeq \;\; C(g^2) \left(-\beta_0 \ln \frac{k^2}{\kappa^2} \right)^{-\gamma_{00}/\beta_0} + \cdots \; . \tag{4}$$

The essential aspect of Eq.(4) is the exponent $(-\gamma_{00}/\beta_0)$, where β_0 is the coefficient in the β-function $\beta(g^2) = \beta_0 g^4 + \cdots$, and γ_{00} has been defined before. (This exponent is actually the same for all gauges $\alpha \geq 0$). It is assumed that $\beta_0 < 0$ corresponding to asymptotic freedom. Under these circumstances, the superconvergence relation (1) is valid for $\gamma_{00}/\beta_0 \; > \; 0$. With the coefficients given by

$$\begin{aligned} \beta_0 \quad &= \quad (16\pi^2)^{-1}(-3N_C \; + \; N_F) \\ \gamma_{00} \quad &= \quad (16\pi^2)^{-1}(-\frac{3}{2}N_C \; + \; N_F) \; , \end{aligned} \tag{5}$$

this condition corresponds to $N_F < \frac{3}{2}N_C$ [4]. For values of N_F below the point $N_F = \frac{3}{2}N_C$, there is confinement of the gauge quanta associated with the group $SU(N_C)$ as described above. Above this point, there is the interval

$$\frac{3}{2}N_C \; < \; N_F \; < \; 3N_C \; , \tag{6}$$

where the theory still has asymptotic freedom, but there is no constraint on transverse gauge excitations. A detailed study of the dispersion representations for the propagator and for projected propagators indicates that no confinement is expected in this region [2].

The results described above show that the zero point of the anomalous dimension coefficient $\gamma_{00}(N_F)$ is as important for the phase structure of the theory as is that of

the coefficient $\beta_0(N_F)$. Both coefficients are invariant under equivalence transformations of the theory, and hence scheme independent. Although fields without mass terms are considered here, if masses of matter fields are present, one can arrange for mass independent coefficients.
The consequences of superconvergence, which are reported above for $N = 1$ supersymetric theories, were first obtained for non-SUSY theories in [2], and then for the SUSY case in [4], where the phase change at $N_F = 3/2N_C$ is pointed out explicitely. In the $SU(N_C)$ gauge theory without supersymmetry, the interval corresponding to Eq. (6) is given by

$$\frac{13}{4}N_C \ < \ N_F \ < \ \frac{22}{4}N_C \ . \tag{7}$$

For $N_F < \frac{13}{4}N_C$, the superconvergence relation is valid, and hence transverse gauge field quanta are confined in the sense that they are not in the physical state space. One may expect that the known perturbative zero ${g_0}^2$ of the renormalization group function $\beta(g^2)$, at the upper end of the interval (6) or (7) respectively, actually corresponds to a general non-pertubative infrared fixed point in the corresponding window [5].

It is very interesting to compare the phase structure obtained from superconvergence with the results following from electric-magnetic DUALITY of $N = 1$ SUSY gauge theories [6]. As above, the electric theory is considered to have the gauge group $G = SU(N_C)$ and massles matter fields in the representation $N_F \times (\mathbf{N_C} + \overline{\mathbf{N}}_\mathbf{C})$. It is proposed that there exists a dual magnetic theory which provides an equivalent description in the infrared. For appropriate values of N_C and N_F, it has the gauge group $G^d = SU(N_C^d)$ with $N_C^d = N_F - N_C$. There are $N_F^d = N_F$ quark superfields q in the fundamental representation, the corresponding anti-quark superfields $\overline{q}$, and N_F^2 independent scalar superfields M, which are coupled via a Yukawa superpotential of the form $\sqrt{\lambda}M_j^i q_i \overline{q}^j$. This coupling is required by the anomaly matching conditions, which are used in the construction of dual theories. In the conformal window, the potential drives the magnetic theory to a fixed point which is the same as the one for the electric theory.

In order to have a dual magnetic theory with a single coupling parameter, one can use the method of the REDUCTION OF COUPLINGS, which is a consequence of the renormalization group equations [7]. For the dual magnetic theory discussed here, the reduction equations have a unique solution with an asymptotic power series

expansion. This solution expresses the Yukawa coupling λ in terms of the magnetic gauge coupling: $\lambda({g_d}^2) = f_0(N_F, N_C){g_d}^2$, where the coefficient f_0 is give explicitly.
In the reduced, as well as in the two-coupling form of the magnetic theory, there is no contribution from the superpotential to the one-loop coefficients β_0^d of the β^d function and γ_{00}^d of the anomalous dimension γ^d for the magnetic gauge field. These coefficients are given by

$$\begin{aligned} \beta_0^d &= (16\pi^2)^{-1}(-2N_F \ + \ 3N_C) \\ \gamma_{00}^d &= (16\pi^2)^{-1}(-\frac{1}{2}N_F \ + \ \frac{3}{2}N_C) \ , \end{aligned} \tag{8}$$

where $N_F^d = N_F$ has been used in order to evaluate both theories at the same number of flavors. However, here these flavors refer to the magnetic gauge group $G^d = SU(N_C^d)$.
Comparing the coefficients for the magnetic theory with those for the corresponding electric theory as given in eq.(5), there emerge the DUALITY RELATIONS [8]

$$\begin{aligned} -\,2\gamma_{00}(N_F) &= \beta_0^d(N_F) \ , \\ -2\gamma_{00}^d(N_F) &= \beta_0(N_F) \ , \end{aligned} \tag{9}$$

with N_F on both sides again referring to the different gauge groups. The factor two is due to the definition of the anomalous dimension used. These duality relations explicitly verify the result obtained from superconvergence, namely that the anomalous dimension coefficient $\gamma_{00}(N_F)$ is of direct importance for the phase structure. With the opposite sign, it exactly coincides with the β function coefficient of the dual magnetic theory. The superconvergence arguments imply that, as a consequence of the change of sign of $\gamma_{00}(N_F)$, the electric theory is in a different phase for $N_F < \frac{3}{2}N_C$ than inside the window (6). With duality, the magnetic description changes from asymptotic freedom inside the window to infrared freedom for $N_F < \frac{3}{2}N_C$. Below the zero of $\beta_0^d(N_F)$, the excitations associated with the magnetic gauge group $SU(N_F - N_C)$ describe the spectrum for $N_F > N_C + 1$, assuming $N_C > 4$. They may be considered as composites of the confined electric quanta. Near $N_F = 3N_C$, the upper end of the window, the rôles of electric and magnetic theories are interchanged . With $\gamma_{00}^d/\beta_0^d \ > \ 0$ and asymptotic freedom, the superconvergence argument implies that for $N_F > 3N_C$ the magnetic gauge quanta

are not in the physical state space of the magnetic theory. The infrared free electric quanta describe the spectrum at low energies, assuming some embedding of the electric theory at small distances, for instance into a string theory. It should be mentioned that the lower limit of the conformal window follows also from the superconformal algebra, given the presence of the non-trivial fixed point.

The duality relations (9) can be shown to be valid for SUSY theories with other gauge groups, and matter fields in the fundamental representation. For example, electric theories with the groups $SO(N_C)$, $Sp(2N_C)$, G_2 and the corresponding dual magnetic theories with $SO(N_F - N_C + 4)$, $Sp(2N_C - 2N_C - 4)$, $SU(N_F - 4)$ have been considered. The relations (9) are usually not valid for theories where matter fields in the adjoint representation are present. In order to have matching anomalies in these cases, it turns out that superpotentials are needed for the electric as well as for the magnetic theories [9]. These potentials then dominate the phase structure, and the gauge coupling plays a lesser rôle. Even though the duality relations (9) may not hold, the superconvergence arguments themselves are still of interest. In particular, superconvergence may give some information about phases of these theories without superpotentials, where there is no duality.

BIBLIOGRAPHY.

[1] R. Oehme, W. Zimmermann, Phys. Rev. **D21** (1980) 471, 1661; R. Oehme, Phys. Lett. **B252** (1990) 641; R. Oehme, W. Xu, Phys. Lett. **B333** (1994) 172, hep-th/9406081, **B384** (1996) 269, hep-th/9604021..

[2] R. Oehme, Phys. Rev. **D42** (1990) 4209, Phys. Lett. **B195** (1987) 60; K. Nishijima, Prog.Theor.Phys. **75** (1986) 1221.

[3] K. Nishijima, Prog.Theor.Phys.**77** (1987) 1053; R. Oehme, Phys. Lett. **B232** (1989) 489; T. Kugo, I. Ojima, Prog.Theor.Phys.Suppl. **66** (1979) 1.

[4] R. Oehme, 'Superconvergence, Supersymmetry and Conformal Invariance', in Leite Lopes Festschrift, pp. 443-457, World Scientific 1988, (KEK Library Scan, HEP SPIRES SLAC: Oehme, R.).

[5] T. Banks, A. Zaks, Nucl. Phys. **B196** (1982) 189; E. Gardi, G. Grunberg, JHEP **024** (1999) 9903, hep-th/9810192.

[6] N. Seiberg, Nucl. Phys. **B435** (1995) 129, hep-th/9411149.

[7] R. Oehme, Phys. Rev. **D59** (1999) 105004-1, hep-th/9808054, 'Reduction of Coupling Parameters in Quantum Field Theories', in Concise Encyclopedia of SUPERSYMMETRY, Kluver 2001, hep-th/0012162, Prog. Theor. Phys. Suppl. **86** (1986) 215; R. Oehme, W. Zimmermann, Commun.Math.Phys. **97** (1985) 569; R. Oehme, K. Sibold and W. Zimmermann, Phys. Lett. **B147** (1984) 115, Phys. Lett. **B153**(1985) 142; W. Zimmermann, Commun. Math. Phys. **97** (1985) 211.

[8] R. Oehme, Phys. Lett. **B399** (1997) 67, hep-th/9701012, 'Superconvergence, Confinement and Duality', Novy Svit, Crimea, 1995: Proceedings pp. 107-116, Kiev 1995, hep-th/9511014.

[9] D. Kutasov, A. Schwimmer, Phys.Lett. **B354** (1995) 315, hep-th/9505004; M. Tachibana, Phys. Rev. **D58** (1998) 045015, hep-th/9709186.

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